Existence of Gradient Kähler-Ricci Solitons

Huai-Dong Cao*

§1. Introduction

We consider the Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} \tag{1.1}$$

on a complete noncompact n-dimensional Kähler manifold M^n. A solution $g_{i\bar{j}}(t)$, for $t \geq 0$, is called a Kähler-Ricci soliton if it moves along Eq.(1.1) under a one-parameter family of biholomorphisms. If this comes from a holomorphic vector field $V = (V^i)$ then we have a soliton when

$$V_{i,\bar{j}} + V_{\bar{j},i} = R_{i\bar{j}},$$

since the metric changes by its Lie derivatives. When the holomorphic vector field is the gradient of a function we say we have a gradient soliton. In this case, we have

$$R_{i\bar{j}} = f_{,i\bar{j}} \qquad \text{and} \qquad f_{,ij} = 0 \tag{1.2}$$

for some real-valued function f on M. We remark that the condition $f_{,ij} = 0$ is equivalent to saying that the vector field $V = \bigtriangledown f$ is holomorphic.

On a compact Kähler manifold M^n with positive first Chern class $C_1(M)$, we can consider the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}} + g_{i\bar{j}} \tag{1.3}$$

A gradient Kähler-Ricci soliton of Eq.(1.3) is then characterized by the equations

$$R_{i\bar{j}} - g_{i\bar{j}} = f_{,i\bar{j}} \qquad \text{and} \qquad f_{,ij} = 0. \tag{1.4}$$

The notion of Ricci soliton was first introduced by Hamilton [4]. It has played a very important role in the study of Ricci flow. One of the important aspects is its relation with the Harnack estimates (see [3] and [5]). Another one is the fact that Ricci solitons often arise as the limits of singularities in the Ricci flow (see [6]). However, few examples were known. In the noncompact case, Hamilton [4] wrote down the first example of a Ricci soliton, called the cigar soliton, on the complex plane $\mathbf{C}$. It has the form

$$ds^2 = \frac{|dz|^2}{1 + |z|^2}, \qquad z \in \mathbf{C}$$

which flows towards the origin by conformal dilations and has positive Gaussian curvature. It is asymptotic to a flat cylinder at infinity and has maximal Gaussian curvature at the origin. Later, Robert Bryant [1] found a gradient Ricci soliton on $\mathbf{R}^3$ with positive curvature operator.

Our purpose in this note is to provide examples of Kähler-Ricci solitons.

* Research partially supported by a NSF grant and a Sloan Research Fellowship.

Theorem 1. *For each $n \geq 1$, there exists on $\mathbf{C}^n$ a complete rotationally symmetric gradient Kähler-Ricci soliton of positive sectional curvature. Moreover such a soliton is unique up to scaling and dilation.*

Remark 1: For $n = 1$, the soliton is just the cigar soliton found by Hamilton.

Remark 2: It can be shown that the soliton metric g on C^n ($n \geq 2$) satisfies the following properties: (i) Let ρ denote the distance function from the origin with respect to g, then the volume of the geodesic ball $B_g(0, \rho)$ with respect to the metric g grows like ρ^n; (ii) The scalar curvature R of the metric g decays like $1/\rho$.

One interesting open problem is the following uniqueness question:

Question: Suppose g is a complete gradient Kähler-Ricci soliton of positive curvature on $\mathbf{C}^n$, is it true that g is necessarilly a rotationally symmetric one (hence given by Theorem 1)?

The anwser to the above question is affirmative when $n = 1$. This is because in general we always have a Killing vector field given by JV which gives a S^1 action on the soliton. Here J denotes the complex structure of $\mathbf{C}^n$.

Theorem 2. *The total space X_n of the anticanonical (or canonical) line bundle of the complex projective space $\mathbf{P}^{n-1}$ ($n \geq 2$) admits a complete rotationally symmetric gradient Kähler-Ricci soliton.*

Remark 3: For $n = 2$, the manifold X_2 is the tangent bundle to the sphere $S^2 = \mathbf{P}^1$ on which Eguch and Hanson [10] constructed the well-known Hyper-Kähler (Calabi-Yau) metric. The soliton metric restricted to each fiber is quasi-isometric to the cigar soliton on the complex plane $\mathbf{C}$. Moreover, Calabi [9] constructed Hyper-Kähler metrics on X_n for $n \geq 3$.

In the compact case, it is known that on Riemann surfaces there are no Ricci solitons except those of constant curvature. Tom Ivey [7] showed that there are no Ricci solitons other than constant curvature metrics on compact 3-dimensional Riemannian manifolds. This naturally leads to the question of the existence of nontrivial Ricci solitons on compact manifolds. On a compact Kähler manifold with positive first Chern class, the nonvanishing of the Futaki invariant is a necessary condition for the existence of a gradient Kähler-Ricci soliton which is not Käler-Einstein. In this article we also construct gradient Kähler-Ricci solitons on certain compact Kähler manifolds:

Theorem 3. *There exists a unique $U(n)$-invariant (shrinking) Kähler-Ricci soliton on the total space M_k of the projective line bundle $\mathbf{P}(L^k \bigoplus L^{-k}) \xrightarrow{\pi} \mathbf{P^{n-1}}$ for each positive integer $1 \leq k \leq n-1$.*

After we finished the construction at Columbia University, we learned from Professor S. Bando, while he was visiting Courant Institute in 1991, that the existence of gradient solitons on these compact manifolds had already been shown by Koiso [8]. Our treatment here, however, is more explicit. Moreover, we show that the soliton metric on M_1 has positive Ricci curvature.

In general, the Ricci soliton equations (1.2) and (1.4) are nonlinear systems and very difficult to solve. In fact, all known examples of soliton metrics so far are rotationally symmetric ones and both (1.2) and (1.4) are reduced to certain nonlinear ODEs. The gradient solitons in Theorem 1 and Theorem 2 are found by solving these ODEs.

We are very grateful to Professor Richard Hamilton for bringing these problems to our attention. We would like to thank Professor Gang Tian for very helpful conversations and for his interest. We also like to thank the Institute for Advanced Study for its support, where part of the work was carried out. Finally, we want to thank the referee for his careful reading and helpful suggestions.

§2. The Gradient Soliton on $\mathbf{C}^n$

In this section we consider rotationally symmetric Käkhler metrics on $\mathbf{C^n}$ and derive the Ricci soliton equation for such metrics.

Any Kähler metric $g_{i\bar{j}}dz^i dz^{\bar{j}}$ on $\mathbf{C^n}$ invariant under the unitary group $U(n)$ can be gererated by a Kähler potential $\Phi(z,\bar{z})$:

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} \Phi(z,\bar{z}) \tag{2.1}$$

such that

$$\Phi(z,\bar{z}) = w(|z|^2), \qquad z \in \mathbf{C^n}$$

for some smooth function $w(r)$.

In fact it is more convenient if we use the variable $t = \log|z|^2$. Then we can express the Kähler potential as

$$\Phi(z,\bar{z}) = u(t), \qquad \big(t = \log|z|^2\big) \tag{2.2}$$

where $u(t)$ is a smooth function on $(-\infty,\infty)$ and as $t \to -\infty$, it has an expansion

$$u(t) = a_0 + a_1 e^t + a_2 e^{2t} + \cdots, \qquad a_1 > 0. \tag{2.3}$$

Conversely, any smooth function $u(t)$ on $(-\infty,\infty)$ generates a $U(n)$-invariant Kähler metric on $\mathbf{C^n}$ if and only if it satisfies the differential inequalities

$$u'(t) > 0, \quad u''(t) > 0, \qquad t \in (-\infty,\infty) \tag{2.4}$$

and the asymptotic condition (2.3).

By (2.1) and (2.2) we have, for $t = \log|z|^2$,

$$g_{i\bar{j}} = \partial_i \partial_{\bar{j}} u(t) = e^{-t} u'(t) \delta_{i\bar{j}} + e^{-2t} \bar{z}_i z_j \big(u''(t) - u'(t)\big). \tag{2.5}$$

Consequently,

$$g^{i\bar{j}} = e^t (u'(t))^{-1} \delta^{i\bar{j}} + z_i \bar{z}_j \big(u''(t)^{-1} - u'(t)^{-1}\big) \tag{2.6}$$

and

$$\det(g_{i\bar{j}}) = e^{-nt}(u'(t))^{n-1}u''(t) \tag{2.7}$$

Let

$$f(t) = -\log\det(g_{i\bar{j}}) = nt - (n-1)\log u'(t) - \log u''(t) \tag{2.8}$$

Then the Ricci tensor of the metric $g_{i\bar{j}}$ is

$$R_{i\bar{j}} = \partial_i \partial_{\bar{j}} f(t) \tag{2.9}$$

Consider the gradient vector field $V^i = g^{i\bar{j}} f_{,\bar{j}}$. Using (2.6) we get

$$V^i = g^{i\bar{j}} e^{-t} z_j f'(t) = z_i \frac{f'(t)}{u''(t)} \tag{2.10}$$

Since z_i is holomorphic and $f'(t)/u''(t)$ is real valued, we see that the vector field V is holomorphic if and only if

$$\frac{f'(t)}{u''(t)} = \alpha$$

for some constant α. We note from (2.10) that the gradient vector field V vanishes at the origin. The soliton flows along $-V$ and hence we should expect α to be negative so that everything is flowing towards the origin as in the case of the cigar soliton when $n = 1$.

It follows that

$$f(t) = \alpha u'(t) + k \tag{2.11}$$

for some constant k.

Plugging (2.8) into (2.11), we derive the following second order equation in u:

$$(u')^{n-1} u'' e^{\alpha u'} = \beta e^{nt} \tag{2.12}$$

where β is some positive constant.

Setting $\phi(t) = u'(t)$, Eq.(2.12) becomes

$$\phi^{n-1}\phi' e^{\alpha\phi} = \beta e^{nt}$$

which is a first order equation in ϕ.

We remark that $\alpha = 0$ corresponds to the flat Euclidean metric. To get nontrivial Ricci solitons we have to require $\alpha > 0$. By scaling, we may assume $\alpha = 1$. In the mean time, we can also normalize $\beta = 1$ by an appropriate translation in t (i.e., a dilation in z). Therefore we only need to consider the equation

$$\phi^{n-1}\phi' e^{\phi} = e^{nt} \tag{2.13}$$

The separation of the variables ϕ and t yields

$$\phi^{n-1} e^{\phi} d\phi = e^{nt} dt$$

Integrating both sides, we get

$$\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\phi^k e^\phi = e^{nt} + C$$

In order that $\phi \to 0$ as $t \to -\infty$ we must have

$$C = (-1)^{n-1}n!$$

So the solution $\phi(t)$ is implicitly given by

$$\sum_{k=0}^{n-1}(-1)^{n-k-1}\frac{n!}{k!}\phi^k e^\phi = e^{nt} + (-1)^{n-1}n! \tag{2.14}$$

One can check from (2.14) that the solution ϕ satisfies the required asymptotic condition (2.3):

$$\phi(t) = a_1 e^t + a_2 e^{2t} + \cdots, \qquad a_1 = 1, \tag{2.15}$$

and the differential inequalities (2.4):

$$\phi(t) > 0, \quad \phi'(t) > 0, \qquad \text{for } t \in (-\infty, \infty) \tag{2.16}$$

Thus it gives rise to a $U(n)$-invariant Kähler metric g on $\mathbf{C^n}$, which is a Ricci soliton. From (2.14), or Eq.(2.13), it is also easy to see that

$$\lim_{t\to\infty} t^{-1}\phi(t) = n, \qquad \lim_{t\to\infty} \phi'(t) = n \tag{2.17}$$

To see that the soliton metric g is complete, let ρ denote the distance function from the origin with respect to g. Since the metric g is rotationally symmetric, it is clear that straight lines through the origin are geodesics and ρ is a function of t only and is given by

$$\rho(t) = \int_{-\infty}^{t} \sqrt{\phi'(\tau)}d\tau$$

It then follows from (2.15)-(2.17) that $\rho = O(t)$, which implies that the metric g is complete.

In summary, we have proved the following

Proposition 2.1. *For each $n \geq 1$, there exists on $\mathbf{C^n}$ a complete rotationally symmetric Kähler-Ricci soliton. Moreover, such a soliton is unique up to scaling (in the metric) and dilation (in the variable z).*

Note that when $n = 1$, we have $\phi = \log(1 + e^t)$. It then follows from (2.5) that

$$ds^2 = \frac{|dz|^2}{1+e^t} = \frac{|dz|^2}{1+|z|^2},$$

which is the cigar soliton observed by Hamilton.

Next we shall prove that the soliton metric obtained above has positive sectional curvature.

In general, the curvature tensor of a Kähler metric $g_{i\bar{j}}$ is given by

$$R_{i\bar{j}k\bar{l}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial z^{\bar{l}}} + g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial z^{\bar{l}}}.$$

For a rotationally symmetric metric on $\mathbf{C^n}$, it suffices to compute the curvature at a point $P = (z_1, 0, \cdots, 0)$. From a straightforward computation we get the following:

At point $P = (z_1, 0, \cdots, 0)$ $(z_1 \neq 0)$,

$$\begin{aligned} R_{i\bar{j}k\bar{l}} = e^{-2t}\Big\{ &(\phi - \phi')(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \\ &+ (3\phi' - 2\phi - \phi'')(\delta_{ij}\delta_{kl1} + \delta_{il}\delta_{jk1} + \delta_{jk}\delta_{il1} + \delta_{kl}\delta_{ij1}) \\ &+ (6\phi'' - 11\phi' - \phi''' + 6\phi)\delta_{ijkl1} + \frac{(\phi' - \phi'')^2}{\phi'}\delta_{ijkl1} \\ &+ \frac{(\phi - \phi')^2}{\phi}(\delta_{ij\hat{1}}\delta_{kl1} + \delta_{il\hat{1}}\delta_{jk1} + \delta_{jk\hat{1}}\delta_{il1} + \delta_{kl\hat{1}}\delta_{ij1})\Big\} \end{aligned} \tag{2.18}$$

where δ_{ij1} and δ_{ijkl1} are zero unless all the indices are 1, while $\delta_{ij\hat{1}}$ is zero unless $i = j$ and neither index is 1.

At the origin, the above formula reduces to

$$R_{i\bar{j}k\bar{l}} = -a_2(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk}) \tag{2.19}$$

where a_2 is the coefficient of e^{2t} in (2.3).

Recall that the sectional curvature of the 2-plane spanned by two (real) tangent vectors

$$X = Re\ v^i \frac{\partial}{\partial z_i} \qquad \text{and} \qquad Y = Re\ w^i \frac{\partial}{\partial z_i}$$

is given by

$$||X \wedge Y||^{-2} R_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} - w^i v^{\bar{j}})(w^k v^{\bar{l}} - v^k w^{\bar{l}})$$

where, up to a constant factor,

$$||X \wedge Y||^2 = g_{i\bar{l}} g_{k\bar{j}} \left[(v^i w^{\bar{j}} - w^i v^{\bar{j}})(w^k v^{\bar{l}} - v^k w^{\bar{l}}) + (v^i w^k - w^i v^k)(v^{\bar{l}} w^{\bar{j}} - w^{\bar{l}} v^{\bar{j}})\right]$$

denotes the square of the area of the parallelogram spanned by X and Y. We remark that $||X \wedge Y||^2 = 0$ (i.e., X and Y are colinear) if and only if

$$v^i w^{\bar{j}} - w^i v^{\bar{j}} = 0 \qquad \text{for all } i,\ j, \tag{2.20}$$

Hence the positivity of sectional curvature is equivalent to

$$R_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} - w^i v^{\bar{j}})(w^k v^{\bar{l}} - v^k w^{\bar{l}}) > 0$$

for all pairs of complex numbers (v^i), (w^i) which do not satisfy (2.20). Without loss of generality, we can assume that

$$v^i = 0, \; i \geq 2 \qquad \text{and} \qquad w^i = 0, \; i \geq 3 \tag{2.21}$$

On one hand we have

$$R_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} - w^i v^{\bar{j}})(w^k v^{\bar{l}} - v^k w^{\bar{l}}) = 2R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} - 2ReR_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} v^k w^{\bar{l}})$$

On the other hand, one computes from (2.18) and (2.21) the holomorphic bisectional curvature

$$R_{i\bar{j}k\bar{l}} v^i v^{\bar{j}} w^k w^{\bar{l}} = e^{-2t}\Big\{\big[\frac{(\phi'')^2}{\phi'} - \phi'''\big]|v^1|^2|w^1|^2 + \big[\frac{(\phi')^2}{\phi} - \phi''\big]|v^1 w^2|^2\Big\}$$

and

$$R_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} v^k w^{\bar{l}}) = e^{-2t}\big[\frac{(\phi'')^2}{\phi'} - \phi'''\big](v^1 w^{\bar{1}})^2$$

Thus

$$\begin{aligned} &R_{i\bar{j}k\bar{l}}(v^i w^{\bar{j}} - w^i v^{\bar{j}})(w^k v^{\bar{l}} - v^k w^{\bar{l}}) \\ &\quad = e^{-2t}\Big\{\big[\frac{(\phi'')^2}{\phi'} - \phi'''\big]|v^1 w^{\bar{1}} - w^1 v^{\bar{1}}|^2 + \big[\frac{(\phi')^2}{\phi} - \phi''\big]|v^1 w^2|^2\Big\} \end{aligned}$$

Therefore the positivity of sectional curvature follows from the following

Lemma 2.2. *Let ϕ be the solution of Eq.(2.13), given by (2.14). Then we have, for all $t \in (-\infty, \infty)$,*

$$\begin{aligned} &\textit{(i)} && \phi - \phi' > 0 \\ &\textit{(ii)} && (\phi')^2 - \phi\phi'' > 0 \\ &\textit{(iii)} && (\phi'')^2 - \phi'\phi''' > 0. \end{aligned}$$

Proof. First we note that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). For example, (ii) implies that the function ϕ'/ϕ is a strictly decreasing function of t hence (i) follows, since $\phi'/\phi = 1$ at $t = -\infty$.

We know that ϕ satisfies the equation

$$\phi' = \frac{e^{nt}}{\phi^{n-1} e^{\phi}} \tag{2.22}$$

On the other hand, since $\phi'(t) > 0$ on $(-\infty, \infty)$, t can be considered as a function of ϕ and we obtain

$$\frac{d}{d\phi}(e^{nt}) = n e^{nt}\frac{dt}{d\phi} = n\phi^{n-1}e^{\phi} \tag{2.23}$$

From (2.22) we compute that

$$\phi'' = n\phi' - (\phi')^2 - (n-1)\frac{(\phi')^2}{\phi}$$

and

$$\phi''' = n^2\phi' - 3n(\phi')^2 + 2(\phi')^3 - 3n(n-1)\frac{(\phi')^2}{\phi} + 4(n-1)\frac{(\phi')^3}{\phi} + (n-1)(2n-1)\frac{(\phi')^2}{\phi^2}$$

Hence

$$\begin{aligned}(\phi'')^2 - \phi'\phi''' =&(\phi')^2\big[n\phi' - (\phi')^2 + n(n-1)\frac{\phi'}{\phi} - 2(n-1)\frac{\phi'^2}{\phi} - n(n-1)\frac{(\phi')^2}{\phi^2}\big]\\ =&\frac{(\phi')^4}{e^{nt}\phi^2}\big[n\phi^{n+1}e^{\phi} - \phi^2 e^{nt} + n(n-1)\phi^n e^{\phi} - 2(n-1)\phi e^{nt} - n(n-1)e^{nt}\big]\end{aligned}$$

Let

$$c(\phi) = n\phi^{n+1}e^{\phi} + n(n-1)\phi^n e^{\phi} - \phi^2 e^{nt} - 2(n-1)\phi e^{nt} - n(n-1)e^{nt} \tag{2.24}$$

Then we have $c(0)=0$. Differentiating (2.24) with respect to ϕ and using (2.23), we get

$$c'(0) = c''(0) = 0$$

and

$$c'''(\phi) = 2n(n-1)\phi^{n-2}e^{\phi}$$

Since $\phi > 0$ on $(-\infty, \infty)$, this implies that $c'''(\phi) > 0$ for all $\phi > 0$. This in turn implies $c''(\phi) > 0$, $c'(\phi) > 0$ and $c(\phi) > 0$. Thus (iii) is proved.

Moreover it follows from Lemma 2.2 (i) that the coefficient $a_2 < 0$, hence (2.19) implies that the sectional curvature at the origin is also positive. Therefore, we have proved that the gradient soliton g on $\mathbf{C}^n$ in Proposition 2.1 has positive sectional curvature. This finishes the proof of Theorem 1.

§3. The Gradient Soliton on the anticanonical bundle of $\mathbf{P}^{n-1}$

In this section we shall consider the total space of the anticanonical (or canonical) bundle over the complex projective space $\mathbf{P}^{n-1}$ and construct a gradient Kähler-Ricci soliton on it. The soliton metric is again a rotationally symmetric one and has nonnegative sectional curvature.

Let $z_1, z_2, \cdots, z_n$ be the coordinate system on complex Euclidean space $\mathbf{C}^n$. The complex projective space $\mathbf{P}^{n-1}$ is the quotient of $\mathbf{C}^n \setminus \{0\}$ by the action of the group of nonzero scalar multiplications. It is covered by n coordinate domains U_j, $j = 1, 2, \cdots, n$, each characterized by $z_j \neq 0$ with the local (inhomogeneous) coordinate system $(z_i/z_j)(1 \leq i \leq n, i \neq j)$. Let L denote the hyperplane bundle over $\mathbf{P}^{n-1}$. For each nonzero integer m, we consider the total space X_m of the line bundle $L^m \xrightarrow{\pi} \mathbf{P^{n-1}}$, where L^m is the mth tensor power of L. Note that X_n is the canonical bundle of $\mathbf{P}^{n-1}$. Without loss of generality, we may choose $m > 0$. The transition functions of the bundle are given by

$$y_i = \big(\frac{z_i}{z_j}\big)^m y_j \tag{3.1}$$

in $\pi^{-1}(U_i \bigcap U_j)$, where $y_i \in \mathbf{C}^1$ is the fiber coordinate in $\pi^{-1}(U_i)$. Let S_0 denote the zero cross section in X_m, given by $y_i = 0$. The complement $X'_m = X_m \setminus (S_0)$ can be globally parametrized by the homogeneous coordinate space $\mathbf{C}^n \setminus \{0\}$ under the m-to-one map

$$(z_1, z_2, \cdots, z_n) \longrightarrow \big((\frac{z_i}{z_j}); (z_j)^m\big) \in X'_m \bigcap \pi^{-1}(U_j) \qquad (z_j \neq 0, 1 \leq i \leq n; i \neq j) \quad (3.2)$$

As in the last section, a $U(n)$-invariant Kähler metric g on $\mathbf{C}^n \setminus \{O\}$ corresponds to a Kähler potential function $\Phi(z, \bar{z})$ given by

$$\Phi(z, \bar{z}) = u(t), \qquad \big(t = \log |z|^2\big)$$

where $u(t)$ is a smooth function on $(-\infty, \infty)$ which satisfies the differential inequalities (2.4). The property that the metric g, pulled back to $X'_m = X_m \setminus (S_0)$, extends to a Kähler metric on all of X_m can be translated into the following asymptotic condition on $u(t)$:

There exists a constant $a > 0$ such that the function $u(t) - at$ has the expansion

$$u(t) - at = a_0 + a_1 e^{mt} + a_2 e^{2mt} + \cdots \quad (3.3)$$

as $t \to -\infty$, with $a_1 > 0$.

Geometrically, the positive constant a times 2π will be the area (in the metric g) of the complex projective line in the zero section $S_0 = \mathbf{P}^{n-1}$.

The same argument as in section 2 (see (2.5)-(2.13)) implies that the soliton metric must satisfy the equation

$$\phi^{n-1} e^{\phi} d\phi = e^{nt} dt \quad (3.4)$$

Hence

$$\sum_{j=0}^{n-1} (-1)^{n-j-1} \frac{n!}{j!} \phi^j e^{\phi} = e^{nt} + C \quad (3.5)$$

In order that $\phi \to a$ as $t \to -\infty$ we must have

$$C = \sum_{j=0}^{n-1} (-1)^{n-j-1} \frac{n!}{j!} a^j e^a. \quad (3.6)$$

Eq.(3.5) also implies that the exponent m in Eq.(3.3) must be equal to n, hence the manifold X_m under consideration has to be the total space of the canonical or anticanonical line bundle of $\mathbf{P}^{n-1}$.

Again from Eq.(3.4), we have

$$\lim_{t \to \infty} t^{-1} \phi(t) = n, \qquad \lim_{t \to \infty} \phi'(t) = n \quad (3.7)$$

which implies the soliton metric on X_n is complete.

The computation of sectional curvature in section 2 shows that away from the zero section S_0 the sectional curvature of the soliton metric on X_n is positive.

Note that when $n = 2$ the manifold X_2 is the tangent bundle over the sphere $\mathbf{P}^1$. In this case one can check that the metric on each fiber has positive Gaussian curvature and is quasi-isometric to a cigar soliton on $\mathbf{C}^1 = \mathbf{R}^2$. Thus the proof of Theorem 2 is completed.

§4. Homothetically Shrinking Gradient Solitons

In this section we shall use a similar argument to construct (homothetically shrinking) gradient Kähler-Ricci solitons of Eq.(1.5) on some compact complex manifolds. These manifolds are ones on which Calabi [2] constructed extremal metrics. They are the total spaces of certain projective line bundles over the complex projective space $\mathbf{P}^{n-1}$. Indeed we have followed closely the notations used in [2] in the previous two sections. Similar constructions can be carried out when the base manifold is any compact symmetric Kähler manifold.

We shall use the same notations as in section 3. Let L again denote the hyperplane bundle over the projective space $\mathbf{P}^{n-1}$. For each nonzero integer k, we consider the total space M_k of the projective line bundle $\mathbf{P}(L^k \bigoplus L^{-k}) \xrightarrow{\pi} \mathbf{P^{n-1}}$. The transition functions of the bundle are given by

$$y_i = \left(\frac{z_i}{z_j}\right)^k y_j$$

in $\pi^{-1}(U_i \bigcap U_j)$, where $y_i \in \mathbf{P^1}$ is the fiber coordinate in $\pi^{-1}(U_i)$. Let S_0 and S_∞ denote the zero and ∞ cross sections in M_k, given by $y_i = 0$ and $y_i = \infty$, respectively. Without loss of generality, we again choose k to be positive. The complement $M_k' = M_k \setminus (S_0 \bigcup S_\infty)$ can be globally parametrized by the homogeneous coordinate space $\mathbf{C}^n \setminus \{0\}$ under the k-to-one map (3.2).

From [2] we know that the maximal compact subgroup G_K of the automorphisms of M_k is isomorphic to $U(n)/Z_k$. A Kähler metric g on M_k invariant under the actions of G_K corresponds to a Kähler metric, again denoted as g, on $\mathbf{C}^n \setminus \{0\}$ generated by a Kähler potential

$$\Phi(z, \bar{z}) = u(t), \qquad \big(t = \log |z|^2\big) \tag{4.1}$$

where $u(t)$ is a smooth function on $(-\infty, \infty)$ which satisfies the differential inequalities (2.4) and the following asymptotic properties:

i) There exists a constant $a > 0$ such that the function $u(t) - at$ has the expansion

$$u(t) - at = a_0 + a_1 e^{kt} + a_2 e^{2kt} + \cdots \tag{4.2}$$

as $t \to -\infty$, with $a_1 > 0$;

ii) There exists a constant $b > 0$ such that the function $u(t) - bt$ has the expansion

$$u(t) - bt = b_0 + b_1 e^{-kt} + b_2 e^{-2kt} + \cdots \tag{4.3}$$

as $t \to \infty$, with $b_1 < 0$.

The positive constants a, b actually specify the Kähler class of the resulting metric. They represent the areas of the two projective lines lying one in each of the two cross

sections S_0 and S_∞. We remark that the Kähler class of the metric g is equal to the first Chern class $C_1(M_k)$ of M_k provided $a = n-k$ and $b = n+k$. So $C_1(M_k)$ is positive precisely when $1 \le k \le n-1$.

For any smooth function $u(t)$, $-\infty < t < \infty$, satisfying (2.4) and the asymptotic conditions (4.2) and (4.3) with $a = n-k$ and $b = n+k$, the formulas (2.5)-(2.10) imply that

$$R_{i\bar{j}} - g_{i\bar{j}} = \partial_i \partial_{\bar{j}}(f-u)$$

where

$$f(t) = -\log\det(g_{i\bar{j}}) = nt - (n-1)\log u'(t) - \log u''(t) \tag{4.4}$$

Hence the gradient vector field of the function $f-u$ is given by

$$V^i = g^{i\bar{j}} e^{-t} z_j (f' - u') = z_i \frac{f'(t) - u'(t)}{u''(t)}$$

so the vector field V is holomorphic if and only if

$$f'(t) - u'(t) = c_1 u''(t) \tag{4.5}$$

for some constant c_1.

Plugging (4.4) into (4.5) and setting $\phi(t) = u'(t)$, we derive the following second order equation in ϕ:

$$\frac{\phi''}{\phi'} + \left[\frac{n-1}{\phi} + c_1\right]\phi' = n - \phi \tag{4.6}$$

Since the variable t does not appear in Eq.(4.6) we can solve for ϕ' and get

$$\begin{aligned}
\phi' =& \frac{1}{\phi^{n-1}e^{c_1\phi}}\left[n\int \phi^{n-1}e^{c_1\phi}d\phi - \int \phi^n e^{c_1\phi}d\phi\right] \\
=& \frac{-1}{c_1^{n+1}\phi^{n-1}}\left[c_1^n\phi^n + \sum_{j=0}^{n-1}(-1)^{n-j}\frac{n!}{j!}(1+c_1)c_1^j\phi^j - c_2 e^{-c_1\phi}\right]
\end{aligned}$$

where c_2 is another constant.

An implicit solution $u(t)$ with $\phi(t) = u'(t)$ is given by

$$t = -\int \frac{c_1^{n+1}\phi^{n-1}d\phi}{c_1^n\phi^n + \sum_{j=0}^{n-1}(-1)^{n-j}\frac{n!}{j!}(1+c_1)c_1^j\phi^j - c_2 e^{-c_1\phi}} \tag{4.7}$$

The asymptotic conditions (4.2) and (4.3) require that the integrand in (4.7) has simple poles at $\phi = n-k$ and $\phi = n+k$ with residues equal to $1/k$ and $-1/k$ respectively. To have a simple pole at $\phi = n-k$ with residue $1/k$, the constants c_1 and c_2 have to satisfy a system of two equations. It turns out that these two equations are identical and given by

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!}(n-k)^{j-1}(n-k-j)c_1^j = c_2 e^{-(n-k)c_1} \tag{4.8}$$

Similarly, the condition of a simple pole at $\phi = n+k$ with residue $-1/k$ corresponds to another equation in c_1 and c_2:

$$\sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!}(n+k)^{j-1}(n+k-j)c_1^j = c_2 e^{-(n+k)c_1} \tag{4.9}$$

Hence, Eq.(4.6) admits a solution, given by (4.7), which satisfies the conditions (4.2) and (4.3) if and only if c_1 and c_2 satisfy both Eq.(4.8) and Eq.(4.9). The latter condition is easily seen to be equivalent to that c_1 is a nonzero root of the equation $h(x) = 0$, where the function $h(x)$ is given by

$$\begin{aligned} h(x) =& e^{2kx}\Big\{\sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!}(n+k)^{j-1}(n+k-j)x^j\Big\} \\ & - \sum_{j=0}^{n}(-1)^{n-j}\frac{n!}{j!}(n-k)^{j-1}(n-k-j)x^j. \end{aligned}$$

Now the existence and uniqueness of the rotationally symmetric Ricci soliton on the manifold $M_k (1 \leq k \leq n-1)$ follows from the following

Lemma 4.1. *The equation $h(x) = 0$ has one and only one nonzero root c_1 with $-1 < c_1 < 0$.*

We shall only outline the proof of Lemma 4.1 and omit many details. It turns out the proof of the uniqueness is kind of interesting.

Step 1: $h^{(i)}(0) = 0$, for $0 \leq i \leq n$.

From direct computations we have, for $0 \leq i \leq n$,

$$\begin{aligned} h^{(i)}(0) =& \sum_{j=0}^{i}\binom{i}{j}(2k)^{i-j}(-1)^{n-j}n!(n+k)^{j-1}(n+k-j) \\ & - (-1)^{n-i}n!(n-k)^{i-1}(n-k-i) \\ =& (-1)^{n-i}n!\big[(n-k)^i - i(n-k)^{i-1} - (n-k)^{i-1}(n-k-i)\big] = 0 \end{aligned}$$

Step 2: $h^{(n+1)}(x) > 0$, for $x \geq 0$.

From direct computations we have

$$\begin{aligned} h^{(n+1)}(x) =& e^{2kx}\sum_{j=0}^{n}\binom{n+1}{j}(2k)^{n+1-j}\sum_{i=0}^{n-j}(-1)^{n-i-j}\frac{n!}{i!}(n+k)^{i+j-1}(n+k-i-j)x^i \\ =& e^{2kx}\sum_{i=0}^{n}C_i x^i \end{aligned}$$

where the coefficients $C_i, i = 0, 1, ..., n$, are given by

$$C_i = \frac{n!}{i!}(n+k)^i\Big[\sum_{j=0}^{n-i}\binom{n+1}{j}(-1)^{n-i-j}(2k)^{n+1-j}(n+k)^{j-1}(n+k-i-j)\Big].$$

We claim that $C_i > 0$ for $0 \le i \le n$. To see this, let

$$B_j = \binom{n+1}{j}(2k)^{n+1-j}(n+k)^{j-1}(n+k-i-j).$$

Then it is easy to show that

$$B_{j+1} > B_j, \qquad \text{for } 0 \le j \le n-i-1 \tag{4.11}$$

Note that for each i, C_i is an alternating sum starting with a positive term when $j = n-i$. Therefore (4.11) implies that $C_i > 0$.

Step 3: The sign of $h(-1)$ is $(-1)^n$.

From direct computations we have

$$\begin{aligned} h(-1) =& (-1)^n n!\Big[e^{-2k}\sum_{j=0}^{n}\frac{1}{j!}(n+k)^{j-1}(n+k-j) - \sum_{j=0}^{n}\frac{1}{j!}(n-k)^{j-1}(n-k-j)\Big] \\ =& (-1)^n\left[e^{-2k}(n+k)^n - (n-k)^n\right]. \end{aligned}$$

Now it can be shown that

$$e^{-2k}(n+k)^n > (n-k)^n, \qquad \text{for } 1 \le k \le n-1.$$

It follows from Step 1–Step 3 that h has no zero in $(0,\infty)$ and has a zero in the interval $(-1,0)$.

Step 4: h has only one zero in the interval $(-\infty,0)$.

Let $y = -x$, then $h(x) = (-1)^{n+1}e^{-2ky}g(y)$ with

$$g(y) = e^{2ky}\Big\{\sum_{j=0}^{n}\frac{n!}{j!}(n-k)^{j-1}(n-k-j)y^j\Big\} - \sum_{j=0}^{n}\frac{n!}{j!}(n+k)^{j-1}(n+k-j)y^j. \tag{4.12}$$

It is then equivalent to show that g only has one zero in $(0,\infty)$. The crucial point here is to observe that on the interval $(0,\infty)$, g can be written as a power series of the following special form:

$$g(y) = \sum_{j=n+1}^{n+l} a_j y^j - \sum_{j=n+l+1}^{\infty} a_j y^j \tag{4.13}$$

where $a_j > 0$ for all j and $l \ge 2$ is some positive integer. Note that for any function g of the type (4.13), the number of sign changes of g in the interval $(0,\infty)$ must agree with the

number of sign changes of its derivatives $g^{(k)}, 1 \leq k \leq n+l$. Hence it can have only one zero in $(0, \infty)$.

To see that g has the form in (4.13), we compute the Taylor expansion of g:

$$g(y) = n! \sum_{i=n+1}^{\infty} b_i y^i = n! \sum_{i=n+1}^{\infty} \Big\{ \sum_{j=0}^{n} \frac{(2k)^{i-j}(n-k)^{j-1}}{(i-j)!j!}(n-k-j) \Big\} y^i.$$

It can be shown that $\{b_i\}$ is a decreasing sequence. Also b_1 must be positive yet b_i cannot all be positive because $g(y)$ does have a zero in $(0, \infty)$. Hence the Taylor expansion of g must have the form stated in (4.13). Thus the proof of Lemma 4.1 is completed.

Finally we mention that the Ricci curvature of the soliton metric g on X_k is positive if and only if $k = 1$. From (2.9) we have

$$R_{i\bar{j}} = e^{-t} f'(t) \delta_{i\bar{j}} + e^{-2t} \bar{z}_i z_j \big(f''(t) - f'(t) \big). \tag{4.10}$$

Hence $R_{i\bar{j}} > 0$ if and only if $f' > 0$ and $f'' > 0$, or by (4.5), if and only if $\phi + c_1 \phi' > 0$ and $\phi' + c_1 \phi'' > 0$. While the first inequality is true for $1 \leq k \leq n-1$, the second one holds only when $k = 1$.

Notes added for the arXiv posting: This is the original paper appeared in the book "Elliptic and Parabolic Methods in Geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, (1996)" (p.1-16), except with the following modifications:

1. Corrected an inaccuracy in the statement of Theorem 2 by removing the phrase "non-negativity of sectional curvature" (of the soliton metric);
2. Modified Remark 3 and added references [9] and [10];
3. Added Theorem 3 in the introduction to summarize the main result in Section 4.

References

[1] Bryant, R.: Existence of a gradient Ricci soliton in dimension three. Preprint.

[2] Calabi, E.: Extremal Kähler metrics. Seminars on Differentail Geometry (S.-T. Yau ed.), Princeton Univ. Press & Univ. of Tokyo Press, Princeton, New York, 1982, pp. 259-290

[3] Cao, H.-D.: On Harnack's inequalities for the Kähler-Ricci flow. Invent. Math. **109**, 1993, pp. 247-263

[4] Hamilton, R.S.: The Ricci flow on surfaces. in Mathematics and General Relativity, Contemporary Mathematics **71**, 1986.

[5] Hamilton, R.S.: The Harnack estimate for the Ricci flow. J. Differ.Geom. **37**, 1993, pp. 225-243

[6] Hamilton, R.S.: Eternal solutions to the Ricci flow. Preprint

[7] Ivey, T.: Ricci solitons on Compact three-manifolds. Differ. Geom. and its Appl. **3**, 1993, pp. 301-307

[8] Koiso, N: On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics. Advanced Studies in Pure Mathematics 18-I, 1990, Recent Topics in Differential and Analytic Geometry, pp. 327-337

[9] Calabi, E.: Métriques kählériennes et fibrés olomorphes, Ann. Sci. École Norm. Sup., **12** (1979), 269–294

[10] Eguchi, T. and Hanson, A. J.: Asymptotically flat selfdual solutions to Euclidean gravity, Phys. Lett. B **74** (1978), 249–251

Department of Mathematics

Texas A&M University

College Station, TX 77843

Email: cao@math.tamu.edu